INFORMATION GEOMETRIC COMPLEXITY OF ENTROPIC MOTION ON CURVED STATISTICAL MANIFOLDS

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ABSTRACT

Physical systems behave according to their underlying dynamical equations which, in turn, can be identified from experimental data. Explaining data requires selecting mathematical models that best capture the data regularities. Identifying dynamical equations from the available data and statistical model selection are both very difficult tasks. Motivated by these fundamental links among physical systems, dynamical equations, experimental data and statistical modeling, we discuss in this invited Contribution our information geometric measure of complexity of geodesic paths on curved statistical manifolds underlying the entropic dynamics of classical physical systems described by probability distributions. We also provide several illustrative examples of entropic dynamical models used to infer macroscopic predictions when only partial knowledge of the microscopic nature of the system is available. Finally, we present entropic arguments to briefly address complexity softening effects due to statistical embedding procedures.

INTRODUCTION

The intimate connection between dynamics, on the one hand, and modeling, prediction, and complexity, on the other, is quite remarkable in science [1]. In real-world experiments, we usually gather data of the state of a physical system at various points in space and time. Then, to achieve some comprehension of the physics behind the behaviour of the system, we must reconstruct the underlying dynamical equations from the data. Deducing dynamics from experimental observations (data) is a fundamental part of science [2], [3]. We observe the trajectories of planets to deduce the laws of celestial mechanics; we consider monetary parameters to determine economic laws; we observe atoms to deduce quantum mechanics. A current challenge is the analysis of data gathered from networks of interferometric gravitational-wave detectors to search for a stochastic gravitational-wave background [4].

A very recent and extremely interesting work shows that deducing the underlying dynamical equations from experimental data is NP hard (the NP complexity class denotes a class of problems that have solutions which can be quickly checked on a classical computer) and is computationally intractable [5]. This hardness result holds true for both classical and quantum systems, and regardless of how much experimental data we gather about the system. These results imply that various closely related problems, such as finding the dynamical equation that best approximates the data, or testing a dynamical model against experimental data, are intractable in general.

By analyzing the available data about a system of interest, it is possible to identify classes of regularities of the system itself. It is generally agreed that something almost entirely random, with practically no regularities, would have an effective complexity near zero [6]. Instead, structured systems (where correlations among system's constituents arise) can be very complex. Structure and correlation are not completely independent of randomness. Indeed, both maximally random and perfectly ordered systems possess no structure [7], [8]. What then is the meaning of complexity? It appears that:

- A good measure of complexity is best justified through utility in further application [9];
- A good measure of complexity is most useful for comparison between things, at least one of which, has high complexity by that measure [6];
- A good measure of complexity for many-body systems ought to obey the so-called slow law growth [10]: complexity ought not to increase quickly, except with low probability, but can increase slowly;
- A good measure of complexity is one for which the motivations for its introduction and the features it is intended to capture are stated in a clear manner [7].

In general, good measures of complexity are introduced within formulations that deal with the whole sequence of events that lead to the object whose complexity is being described [9]. For such measures, that which is reached only through a difficult path is complex. For instance, when defining the complexity of a noisy quantum channel, the concept of pattern plays a role, in some sense [11]. The thermodynamic and the logical depths are two such measures as well. The thermodynamic depth is the measure of complexity proposed by Lloyd and Pagels and it represents the amount of entropy produced during a state's actual evolution [12]. The logical depth is a measure of complexity proposed by Bennett and it represents the execution time required for a universal Turing machine to run the minimal program that reproduces (say) a system's configuration [13].

Since the path leading to an object (or, state) is central when defining a measure of complexity, simple thermodynamic criteria applied to the states to be compared are inadequate. Thermodynamic potentials measure a system's capacity for irreversible change, but do not agree with intuitive notions of complexity [10]. For instance, the thermodynamic entropy, a measure of randomness, is a monotonic function of temperature where high (low) temperature corresponds to high (low) randomness. However, given that there are many functions that vanish in the extreme ordered and disordered limits, it is clear that requiring this property does not sufficiently restrict a complexity measure of statistical nature (statistical complexity [8] is a quantity that measures the amount of memory needed, on average, to statistically reproduce a given configuration). Despite these facts, it is undisputable that thermodynamics does play a key role when investigating qualitative differences in the complexity of reversible and dissipative systems [13].

The difficulty of constructing a good theory from a data set can be roughly identified with cripticity while the difficulty of making predictions from the theory can be regarded as a rough interpretation of logical depth. Both cripticity and logical depth are intimately related to the concept of complexity. Making predictions can be very difficult in general, especially in composite systems where interactions between subsystems are introduced. The introduction of interactions leads to fluctuation growth which, in turn, can cause the dynamics to become nonlinear and chaotic. Such phenomena are very common and can occur in both natural (cluster of stars) and artificial (financial network) complex dynamical systems [14]. A fundamental problem in the physics of complex systems is model reduction, that is finding a low-dimensional model that captures the gross features of a high-dimensional system [15]. Sometimes, to make reliable macroscopic predictions, considering the dynamics alone may not be sufficient and entropic arguments should be taken into account as well [16].

As stated earlier, one of the major goals of physics is modelling and predicting natural phenomena using relevant information about the system of interest. Taking this statement seriously, it is reasonable to expect that the laws of physics should reflect the methods for manipulating information. This point of view constitutes quite a departure from the conventional line of thinking where laws of physics are used to manipulate information. For instance, in quantum information science, information is manipulated using the laws of quantum mechanics. This alternative perspective is best represented in the so-called Entropic Dynamics (ED) [17], a theoretical framework built on both maximum relative entropy (MrE) methods [18] and information geometric techniques [19]. The most intriguing question being pursued in ED stems from the possibility of deriving dynamics from purely entropic arguments. Indeed, the ED approach has already been applied for the derivation of Newton's dynamics [20] and aspects of quantum theory [21].

In this invited Contribution, inspired by the ED approach to physics and motivated by these fundamental links among physical systems, dynamical equations, experimental data and statistical modeling, we present our information geometric measure of complexity of geodesic paths on curved statistical manifolds underlying the entropic dynamics of classical physical systems described by probability distributions. We also provide several illustrative examples of entropic dynamical models used to infer macroscopic predictions when only partial knowledge of the microscopic nature of the system is available. Finally, we emphasize the relevance of entropic arguments in addressing complexity softening effects due to statistical embedding procedures.

COMPLEXITY

In [22], the so-called Information Geometric Approach to Chaos (IGAC) was presented. The IGAC uses the ED formalism to study the complexity of informational geodesic flows on curved statistical manifolds underlying the entropic dynamics of classical physical systems described by probability distributions.

A geodesic on a curved statistical manifold $\mathcal{M}_S$ represents the maximum probability path a complex dynamical system explores in its evolution between initial and final macrostates. Each point of the geodesic is parametrized by the macroscopic dynamical variables $\{\theta\}$ defining the macrostate of the system. Furthermore, each macrostate is in a one-to-one correspondence with the probability distribution $\{p(x|\theta)\}$ representing the maximally probable description of the system being considered. The quantity x is a microstate of the microspace $\mathcal{X}$. The set of macrostates forms the parameter space $\mathcal{D}_\theta$ while the set of probability distributions forms the statistical manifold $\mathcal{M}_S$.

The IGAC is the information geometric analogue of conventional geometrodynamical approaches [23], [24] where the classical configuration space is being replaced by a statistical manifold with the additional possibility of considering chaotic dynamics arising from non conformally flat metrics (the Jacobi metric is always conformally flat, instead). It is an information geometric extension of the Jacobi geometrodynamics (the geometrization of a Hamiltonian system by transforming it to a geodesic flow [25]).

The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of the equation of motion. The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object in which all the available manifest symmetries are retained- the manifold on which geodesic flow is induced. For example, integrability of the system is connected with existence of Killing vectors and tensors on this manifold. The sensitive dependence of trajectories on initial conditions, which is a key ingredient of chaos, can be investigated from the equation of geodesic deviation. In the Riemannian [23] and Finslerian [24] (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, a very challenging problem is finding a rigorous relation among sectional curvatures, Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy [26].

Information metric

An n-dimensional $\mathbb{C}^\infty$ differentiable manifold is a set of points $\mathcal{M}$ admitting coordinate systems $\mathcal{C}_\mathcal{M}$ and satisfies the following two conditions: 1) each element $c \in \mathcal{C}_\mathcal{M}$ is a one-to-one mapping from $\mathcal{M}$ to some open subset of $\mathbb{R}^n$; 2) For all $c \in \mathcal{C}_\mathcal{M}$, given any one-to-one mapping ξ from $\mathcal{M}$ to $\mathbb{R}^n$, we have that $\xi \in \mathcal{C}_\mathcal{M} \Leftrightarrow \xi \circ c^{-1}$ is a $\mathbb{C}^\infty$ diffeomorphism. In this article, the points of $\mathcal{M}$ are probability distributions. Furthermore, we consider Riemannian manifolds $(\mathcal{M}, g)$. The Riemannian metric g is not naturally determined by the structure of $\mathcal{M}$ as a manifold. In principle, it is possible to consider an infinite number of Riemannian metrics on $\mathcal{M}$. A fundamental assumption in the information geometric framework is the choice of the Fisher-Rao information metric as the metric that underlies the Riemannian geometry of probability distributions [19], [27], [28], namely

$$g_{\mu\nu}(\theta) \stackrel{\text{def}}{=} \int dx p(x|\theta)\, \partial_\mu \log p(x|\theta)\, \partial_\nu \log p(x|\theta), \tag{1}$$

with $\mu, \nu = 1,..., n$ for an n-dimensional manifold and $\partial_\mu \stackrel{\text{def}}{=} \frac{\partial}{\partial\theta^\mu}$. The quantity x labels the microstates of the system. The choice of the information metric can be motivated in several ways, the strongest of which is Cencov's characterization theorem [29]. In this theorem, Cencov proves that the information metric is the only Riemannian metric (except for a constant scale factor) that is invariant under a family of probabilistically meaningful mappings termed congruent embeddings by Markov morphism [29], [30].

Given a statistical manifold $\mathcal{M}_S$ with a metric $g_{\mu\nu}$, the ED is concerned with the following issue [17]: given the initial and final states, what trajectory is the system expected to follow? The answer turns out to be that the expected trajectory is the geodesic that passes through the given initial and final states. Furthermore, the trajectory follows from a principle of inference, the MrE method [18]. The objective of the MrE method is to update from a prior distribution q to a posterior distribution $P(x)$ given the information that the posterior lies within a certain family of distributions p. The selected posterior $P(x)$ is that which maximizes the logarithm relative entropy $\mathcal{S}[p|q]$,

$$\mathcal{S}[p|q] \stackrel{\text{def}}{=} -\int dx p(x) \log \frac{p(x)}{q(x)}. \tag{2}$$

Since prior information is valuable, the functional $\mathcal{S}[p|q]$ has been chosen so that rational beliefs are updated only to the extent required by the new information. We emphasize that ED is formally similar to other generally covariant theories: the dynamics is reversible, the trajectories are geodesics, the system supplies its own notion of an intrinsic time, the motion can be derived from a variational principle of the form of Jacobi's action principle rather than the more familiar principle of Hamilton. In short, the canonical Hamiltonian formulation of ED is an example of a constrained information-dynamics where the information-constraints play the role of generators of evolution. For more details on the ED, we refer to [17].

A geodesic on a n-dimensional curved statistical manifold $\mathcal{M}_S$ represents the maximum probability path a complex dynamical system explores in its evolution between initial and final macrostates θ_{initial} and θ_{final}, respectively. Each point of the geodesic represents a macrostate parametrized by the macroscopic dynamical variables $\theta \equiv (\theta^1,..., \theta^n)$ defining the macrostate of the system. Each component θ^k with $k = 1,..., n$ is a solution of the geodesic equation [17],

$$\frac{d^2\theta^k}{d\tau^2} + \Gamma^k_{lm} \frac{d\theta^l}{d\tau} \frac{d\theta^m}{d\tau} = 0. \tag{3}$$

Furthermore, as stated earlier, each macrostate θ is in a one-to-one correspondence with the probability distribution $p(x|\theta)$. This is a distribution of the microstates x.

Entropic motion

The main objective of ED is to derive the expected trajectory of a system, assuming it evolves from a known initial state θ_i to a known final state θ_f. The ED framework implicitly assumes there exists a trajectory, in the sense that, large changes are the result of a continuous succession of very many small changes. Therefore, the problem of studying large changes is reduced to the much simpler problem of studying small changes. Focusing on small changes and assuming that the change in going from the initial state θ_i to the final state $\theta_f = \theta_i + \Delta\theta$ is sufficiently small, the distance Δl between such states becomes,

$$\Delta l^2 \stackrel{\text{def}}{=} g_{\mu\nu}(\theta)\, \Delta\theta^\mu \Delta\theta^\nu. \tag{4}$$

Following Caticha's work in [17], we explain how to determine which states are expected to lie on the expected trajectory between θ_i and θ_f. First, in going from the initial to the final state the system must pass through a halfway point, that is, a state θ that is equidistant from θ_i and θ_f. Upon choosing the halfway state, the expected trajectory of the system can be determined. Indeed, there is nothing special about halfway states. For instance, we could have similarly argued that in going from the initial to the final state the system must first traverse a third of the way, that is, it must pass through a state that is twice as distant from θ_f as it is from θ_i. In general, the system must pass through an intermediate states θ_ξ such that, having already moved a distance dl away from the initial θ_i, there remains a distance ξdl to be covered to reach the final θ_f . Halfway states have $\xi = 1$, third of the way states have $\xi = 2$, and so on. Each different value of ξ provides a different criterion to select the trajectory. If there are several ways to determine a trajectory, consistency requires that all these ways should agree. The selected trajectory must be independent of ξ. Therefore, the main ED problem becomes the following: initially, the system is in state $p(x|\theta_i)$ and new information in the form of constraints is given to us; the system has moved to one of the neighboring states in the family $p(x|\theta_\xi)$; the problem becomes that of selecting the proper $p(x|\theta_\xi)$. This new formulation of the ED problem is precisely the kind of problem to be addressed using the MrE method. We recall that the MrE method is a method for processing information. It allows us to go from an old set of rational beliefs, described by the prior probability distribution, to a new set of rational beliefs, described by the posterior distribution, when the available information is just a specification of the family of distributions from which the posterior must be selected. Usually, this family of posteriors is defined by the known expected values of some relevant variables. It should be noted however, that it is not strictly necessary for the family of posteriors to be defined via expectation values, nor does the information-constraints need to be linear functionals. In ED, constraints are defined geometrically. Whenever one contemplates using the MrE method, it is important to specify which entropy should be maximized. The selection of a distribution $p(x|\theta)$ requires that the entropies to be considered must be of the form,

$$S[p|q] \stackrel{\text{def}}{=} -\int dx p(x|\theta) \log\left(\frac{p(x|\theta)}{q(x)}\right). \tag{5}$$

Equation (5) defines the entropy of $p(x|\theta)$ relative to the prior $q(x)$. The interpretation of $q(x)$ as the prior follows from the logic behind the MrE method itself. The selected posterior distribution should coincide with the prior distribution when there are no constraints. Since the distribution that maximizes $S[p|q]$

subject to no constraints is $p \propto q$, we must set $q(x)$ equal to the prior. That said, let us return to our ED problem. Assuming we know that the system is initially in state $p(x|\theta_i)$ but have obtained no information reflecting that the system has moved. We therefore have no reason to believe that any change has occurred. The prior $q(x)$ should be chosen so that the maximization of $S[p|q]$ subject to no constraints leads to the posterior $p = p(x|\theta_i)$. The correct choice is $q(x) = p(x|\theta_i)$. If on the other hand we know that the system is initially in state $p(x|\theta_i)$ and furthermore, we obtain information that the system has moved to one of the neighboring states in the family $p(x|\theta_\xi)$, then the correct selection of the posterior probability distribution is obtained by maximizing the entropy,

$$S[\theta|\theta_i] \stackrel{\text{def}}{=} -\int dx p(x|\theta) \log\left(\frac{p(x|\theta)}{p(x|\theta_i)}\right), \tag{6}$$

subject to the constraint $\theta = \theta_\xi$. For the sake of reasoning, let us assume that the system evolves from a known initial state θ_i to a known final state $\theta_f = \theta_i + \Delta\theta$. Furthermore, let us denote with $\theta_\xi = \theta_i + d\theta$ ($\xi \in \mathbb{R}_0^+$) an arbitrary intermediate state infinitesimally close to θ_i. Thus, the distance $d(\theta_i, \theta_f) \stackrel{\text{def}}{=} dl^2_{i\to f}$ between θ_i to and θ_f is given by,

$$dl^2_{i\to f} \stackrel{\text{def}}{=} g_{\mu\nu}(\theta)\Delta\theta^\mu\Delta\theta^\nu, \tag{7}$$

while the distance between θ_i to and θ_ξ reads,

$$dl^2_{i\to\xi} \stackrel{\text{def}}{=} g_{\mu\nu}(\theta)\, d\theta^\mu d\theta^\nu. \tag{8}$$

Finally, the distance between θ_ξ and θ_f becomes,

$$dl^2_{\xi\to f} \stackrel{\text{def}}{=} g_{\mu\nu}(\theta)(\Delta\theta^\mu - d\theta^\mu)(\Delta\theta^\nu - d\theta^\nu). \tag{9}$$

The MrE maximization problem is to maximize $S[\theta_\xi|\theta_i] = S[\theta_i + d\theta|\theta_i]$,

$$S[\theta_i + d\theta|\theta_i] \stackrel{\text{def}}{=} -\frac{1}{2} g_{\mu\nu}(\theta)\, d\theta^\mu d\theta^\nu = -\frac{1}{2} dl^2_{i\to\xi}, \tag{10}$$

under variations of $d\theta$ subject to the geometric constraint,

$$\xi dl_{i\to\xi} = dl_{\xi\to f}, \tag{11}$$

or equivalently, $\xi^2 dl^2_{i\to\xi} - dl^2_{\xi\to f} = 0$. It must then be true that,

$$\delta\left[-\frac{1}{2} g_{\mu\nu}(\theta)\, d\theta^\mu d\theta^\nu - \lambda\left(\xi^2 dl^2_{i\to\xi} - dl^2_{\xi\to f}\right)\right] = 0, \tag{12}$$

where λ denotes a Lagrangian multiplier. Substituting Eqs. (8) and (9) into Eq. (12), we obtain

$$\left\{\left[1 + 2\lambda\left(\xi^2 - 1\right)\right] d\theta_\mu + 2\lambda\Delta\theta_\mu\right\}\delta\left(d\theta^\mu\right) = 0. \tag{13}$$

Since (13) must hold for any $\delta(d\theta^\mu)$, it must be the case that

$$\left\{\left[1 + 2\lambda\left(\xi^2 - 1\right)\right] d\theta_\mu + 2\lambda\Delta\theta_\mu\right\} = 0, \tag{14}$$

that is,

$$d\theta_\mu = \chi\Delta\theta_\mu, \tag{15}$$

where $\chi = \chi(\xi, \lambda)$ is defined as,

$$\chi(\xi, \lambda) \stackrel{\text{def}}{=} \frac{1}{(1 - \xi^2) - \frac{1}{2\lambda}}. \tag{16}$$

To find the value of the Lagrange multiplier λ, observe that the geometric constraint in Eq. (11) can be rewritten as, $\xi^2 dl^2_{i\to\xi} - dl^2_{\xi\to f} = 0$. Then, using Eqs. (8), (9) and (15), we obtain

$$\left[\xi^2\chi^2 - (1 - \chi)^2\right] g_{\mu\nu}(\theta)\Delta\theta^\mu\Delta\theta^\nu = 0, \tag{17}$$

thus,

$$\xi^2\chi^2 - (1 - \chi)^2 = 0. \tag{18}$$

Combining Eqs. (16) and (18), we find

$$\chi(\xi) \stackrel{\text{def}}{=} \frac{1}{1 + \xi} \text{ and, } \lambda(\xi) \stackrel{\text{def}}{=} -\frac{1}{2\xi(1 + \xi)}. \tag{19}$$

In conclusion, it has been determined that

$$dl^2_{i\to\xi} \stackrel{\text{def}}{=} \frac{1}{(1 + \xi)^2}\Delta\theta^2, \tag{20}$$

and,

$$dl^2_{\xi\to f} \stackrel{\text{def}}{=} \frac{\xi^2}{(1 + \xi)^2}\Delta\theta^2. \tag{21}$$

From Eqs. (20) and (21), it follows that

$$dl_{i\to\xi} + dl_{\xi\to f} = \frac{1}{1 + \xi}\Delta\theta + \frac{\xi}{1 + \xi}\Delta\theta = \Delta\theta. \tag{22}$$

However, recall that $dl^2_{i\to f} \stackrel{\text{def}}{=} g_{\mu\nu}(\theta)\Delta\theta^\mu\Delta\theta^\nu = \Delta\theta^2$, that is

$$dl_{i\to f} = \Delta\theta. \tag{23}$$

Combining Eqs. (22) and (23), we arrive at

$$dl_{i\to f} = dl_{i\to\xi} + dl_{\xi\to f}. \tag{24}$$

In other words, given

$$\Delta\theta \stackrel{\text{def}}{=} d\theta + (\Delta\theta - d\theta), \tag{25}$$

we have shown by means of entropic arguments that,

$$\|\Delta\theta\| = \|d\theta\| + \|\Delta\theta - d\theta\|, \tag{26}$$

where $\|\Delta\theta\| \stackrel{\text{def}}{=} \sqrt{dl^2_{i\to f}}$, $\|d\theta\| \stackrel{\text{def}}{=} \sqrt{dl^2_{i\to\xi}}$ and, $\|\Delta\theta - d\theta\| \stackrel{\text{def}}{=} \sqrt{dl^2_{\xi\to f}}$. Given Eq. (25), Eq. (26) holds true iff $d\theta$ and $\Delta\theta - d\theta$ are collinear. Therefore, the expected trajectory is a straight line: the triangle defined by the points θ_i, θ_ξ, and θ_f degenerates into a straight line. This is sufficient to determine a short segment of the trajectory: all intermediate states lie on the straight line between θ_i and θ_f. The generalization beyond short trajectories is immediate: if any three nearby points along a curve lie on a straight line the curve is a *geodesic*. This result is independent of the arbitrarily chosen value ξ so the potential consistency problem we mentioned before does not arise. Summarizing, the answer to the ED problem is the following: *the expected trajectory between a known initial and final state is the geodesic that passes through them.* However, the question of whether the actual trajectory is the expected trajectory remains unanswered and depends on whether the information encoded in the initial state is sufficient for prediction.

Volumes in curved statistical manifolds

Once the distances among probability distributions have been assigned using the Fisher-Rao information metric tensor $g_{\mu\nu}(\theta)$, a natural next step is to obtain measures for extended regions in the space of distributions. Consider an n-dimensional volume of the statistical manifold $\mathcal{M}_s$ of distributions $p(x|\theta)$ labelled by parameters θ^μ with $\mu = 1,..., n$. The parameters θ^μ are coordinates for the point p and in these coordinates it may not be obvious how to write an expression for a volume element $d\mathcal{V}_{\mathcal{M}_s}$. However, within a sufficiently small region any curved space looks flat. That is to say, curved spaces are locally flat. The idea then is rather simple: within that very small region, we should use Cartesian coordinates wherein the metric takes a very simple form, namely the identity matrix $\delta_{\mu\nu}$. In locally Cartesian coordinates χ^α the volume element is given by the product $d\mathcal{V}_{\mathcal{M}_s} \stackrel{\text{def}}{=} d\chi^1 d\chi^2 .....d\chi^n$, which in terms of the old coordinates reads,

$$d\mathcal{V}_{\mathcal{M}_s} \stackrel{\text{def}}{=} \left|\frac{\partial\chi}{\partial\theta}\right| d\theta^1 d\theta^2 ...\ d\theta^n. \tag{27}$$

The problem at hand then is the calculation of the Jacobian $\left|\frac{\partial\chi}{\partial\theta}\right|$ of the transformation that takes the metric $g_{\mu\nu}$ into its Euclidean form $\delta_{\mu\nu}$. Let the new coordinates be defined by $\chi^\mu \stackrel{\text{def}}{=} \Xi^\mu\left(\theta^1,...., \theta^n\right)$ where Ξ denotes a coordinates transformation map. A small change $d\theta$ corresponds to a small change $d\chi$,

$$d\chi^\mu \stackrel{\text{def}}{=} X^\mu_m d\theta^m \text{ where } X^\mu_m \stackrel{\text{def}}{=} \frac{\partial\chi^\mu}{\partial\theta^m}, \tag{28}$$

and the Jacobian is given by the determinant of the matrix X^μ_m, $\left|\frac{\partial\chi}{\partial\theta}\right| \stackrel{\text{def}}{=} \left|\det\left(X^\mu_m\right)\right|$. The distance between two neighboring points is the same whether we compute it in terms of the old or the new coordinates, $dl^2 = g_{\mu\nu}d\theta^\mu d\theta^\nu = \delta_{\alpha\beta}d\chi^\alpha d\chi^\beta$. Therefore the relation between the old and the new metric is $g_{\mu\nu} = \delta_{\alpha\beta}X^\alpha_\mu X^\beta_\nu$. Taking the determinant of $g_{\mu\nu}$, we obtain $g \stackrel{\text{def}}{=} \det(g_{\mu\nu}) = \left[\det\left(X^\alpha_\mu\right)\right]^2$ and therefore $\left|\det\left(X^\alpha_\mu\right)\right| = \sqrt{g}$. Finally, we have succeeded in expressing the volume element totally in terms of the coordinates θ and the known metric $g_{\mu\nu}(\theta)$, $dV_{\mathcal{M}_s} \stackrel{\text{def}}{=} \sqrt{g}d^n\theta$. Thus, the volume of any extended region on the manifold is given by,

$$\mathcal{V}_{\mathcal{M}_s} \stackrel{\text{def}}{=} \int d\mathcal{V}_{\mathcal{M}_s} = \int \sqrt{g}d^n\theta. \tag{29}$$

Observe that $\sqrt{g}d^n\theta$ is a scalar quantity and is therefore invariant under orientation preserving general coordinate transformations $\theta \to \theta'$. The square root of the determinant $g(\theta)$ of the metric tensor $g_{\mu\nu}(\theta)$ and the flat infinitesimal volume element $d^n\theta$ transform as,

$$\sqrt{g(\theta)} \stackrel{\theta\to\theta'}{\to} \left|\frac{\partial\theta'}{\partial\theta}\right| \sqrt{g(\theta')},\ d^n\theta \stackrel{\theta\to\theta'}{\to} \left|\frac{\partial\theta}{\partial\theta'}\right| d^n\theta', \tag{30}$$

respectively. Therefore, it follows that

$$\sqrt{g(\theta)}d^n\theta \stackrel{\theta\to\theta'}{\to} \sqrt{g(\theta')}d^n\theta'. \tag{31}$$

Equation (31) implies that the infinitesimal statistical volume element is invariant under general coordinate transformations that preserve orientation (that is, with positive Jacobian). For more details on these aspects, we suggest Caticha's 2012 tutorial [31].

Information geometric complexity

The elements (or points) $\{p(x|\theta)\}$ of an n-dimensional curved statistical manifold $\mathcal{M}_s$ are parametrized using n real valued variables $\left(\theta^1,..., \theta^n\right)$,

$$\mathcal{M}_s \stackrel{\text{def}}{=} \left\{p(x|\theta) : \theta = \left(\theta^1,..., \theta^n\right) \in \mathcal{D}^{(\text{tot})}_\theta\right\}. \tag{32}$$

The set $\mathcal{D}^{(\text{tot})}_\theta$ is the entire parameter space (available to the system) and is a subset of $\mathbb{R}^n$,

$$\mathcal{D}^{(\text{tot})}_\theta \stackrel{\text{def}}{=} \bigotimes_{k=1}^{n} I_{\theta^k} = \left(I_{\theta^1} \otimes I_{\theta^2} ... \otimes I_{\theta^n}\right) \subseteq \mathbb{R}^n \tag{33}$$

where I_{θ^k} is a subset of $\mathbb{R}$ and represents the entire range of allowable values for the macrovariable θ^k. For example, considering the statistical manifold of one-dimensional Gaussian probability distributions parametrized in terms of $\theta = (\mu, \sigma)$, we obtain

$$\mathcal{D}^{(\text{tot})}_\theta \stackrel{\text{def}}{=} I_\mu \otimes I_\sigma = \left[(-\infty, +\infty) \otimes (0, +\infty)\right], \tag{34}$$

with $I_\mu \otimes I_\sigma \subseteq \mathbb{R}^2$. In the IGAC, we are interested in a probabilistic description of the evolution of a given system in terms of its corresponding probability distribution on $\mathcal{M}_s$ which is homeomorphic to $\mathcal{D}^{(\text{tot})}_\Theta$. Assume we are interested in the evolution from τ_{initial} to τ_{final}. Within the present probabilistic description, this is equivalent to studying the shortest path (or, in terms of the MrE methods [18], the maximally probable path) leading from $\theta(\tau_{\text{initial}})$ to $\theta(\tau_{\text{final}})$.

Is there a way to quantify the complexity of such path? We propose the so-called information geometric entropy (IGE) $\mathcal{S}_{\mathcal{M}_s}(\tau)$ as a good complexity quantifier [32]. In what follows, we highlight the key-points leading to the construction of this quantity.

The IGE, an indicator of temporal complexity of geodesic paths within the IGAC framework, is defined as [32],

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \overset{\text{def}}{=} \log \widetilde{vol}\left[\mathcal{D}_\theta(\tau)\right], \tag{35}$$

where the average dynamical statistical volume $\widetilde{vol}\left[\mathcal{D}_\theta(\tau)\right]$ (which we also choose to name the information geometric complexity (IGC)) is given by,

$$\widetilde{vol}\left[\mathcal{D}_\theta(\tau)\right] \overset{\text{def}}{=} \frac{1}{\tau}\int_0^\tau d\tau' vol\left[\mathcal{D}_\theta(\tau')\right]. \tag{36}$$

Note that the tilde symbol in (36) denotes the operation of temporal average. The volume $vol\left[\mathcal{D}_\theta(\tau')\right]$ in the RHS of (36) is given by,

$$vol\left[\mathcal{D}_\theta(\tau')\right] \overset{\text{def}}{=} \int_{\mathcal{D}_\theta(\tau')} \rho_{(\mathcal{M}_s,\, g)}\left(\theta^1,..., \theta^n\right) d^n\theta, \tag{37}$$

where $\rho_{(\mathcal{M}_s,\, g)}\left(\theta^1,..., \theta^n\right)$ is the so-called Fisher density and equals the square root of the determinant of the metric tensor $g_{\mu\nu}(\theta)$ with $\theta \equiv \left(\theta^1,..., \theta^n\right)$,

$$\rho_{(\mathcal{M}_s,\, g)}\left(\theta^1,..., \theta^n\right) \overset{\text{def}}{=} \sqrt{g(\theta)}. \tag{38}$$

The integration space $\mathcal{D}_\theta(\tau')$ in (37) is defined as follows,

$$\mathcal{D}_\theta(\tau') \overset{\text{def}}{=} \left\{\theta : \theta^k(0) \leq \theta^k \leq \theta^k(\tau')\right\}, \tag{39}$$

where $k = 1,.., n$ and $\theta^k \equiv \theta^k(s)$ with $0 \leq s \leq \tau'$ such that,

$$\frac{d^2\theta^k(s)}{ds^2} + \Gamma^k_{lm}\frac{d\theta^l}{ds}\frac{d\theta^m}{ds} = 0. \tag{40}$$

The integration space $\mathcal{D}_\theta(\tau')$ in (39) is an n-dimensional subspace of the whole (permitted) parameter space $\mathcal{D}_\theta^{(\text{tot})}$. The elements of $\mathcal{D}_\theta(\tau')$ are the n-dimensional macrovariables $\{\theta\}$ whose components θ^k are bounded by specified limits of integration $\theta^k(0)$ and $\theta^k(\tau')$ with $k = 1,.., n$. The limits of integration are obtained via integration of the n-dimensional set of coupled nonlinear second order ordinary differential equations characterizing the geodesic equations. Formally, the IGE is defined in terms of a averaged parametric $(n+1)$-fold integral (τ is the parameter) over the multidimensional geodesic paths connecting $\theta(0)$ to $\theta(\tau)$. Further conceptual details about the IGE and the IGC can be found in [33].

APPLICATIONS

In the following, we outline several selected applications concerning the complexity characterization of geodesic paths on curved statistical manifolds within the IGAC framework.

Gaussian statistical models

In [32], [34], we apply the IGAC to study the dynamics of a system with l degrees of freedom, each one described by two pieces of relevant information, its mean expected value and its variance (Gaussian statistical macrostates). This leads to consider a statistical model on a non-maximally symmetric $2l$-dimensional statistical manifold $\mathcal{M}_s$. It is shown that $\mathcal{M}_s$ possesses a constant negative scalar curvature proportional to the number of degrees of freedom of the system, $\mathcal{R}_{\mathcal{M}_s} = -l$. It is found that the system explores statistical volume elements on $\mathcal{M}_s$ at an exponential rate. The information geometric entropy $\mathcal{S}_{\mathcal{M}_s}$ increases linearly in time (statistical evolution parameter) and, moreover, is proportional to the number of degrees of freedom of the system, $\mathcal{S}_{\mathcal{M}_s} \overset{\tau\to\infty}{\sim} l\lambda\tau$ where λ is the maximum positive Lyapunov exponent characterizing the model. The geodesics on $\mathcal{M}_s$ are hyperbolic trajectories. Using the Jacobi-Levi-Civita (JLC) equation for geodesic spread, we show that the Jacobi vector field intensity $J_{\mathcal{M}_s}$ diverges exponentially and is proportional to the number of degrees of freedom of the system, $J_{\mathcal{M}_s} \overset{\tau\to\infty}{\sim} l\exp(\lambda\tau)$. The exponential divergence of the Jacobi vector field intensity $J_{\mathcal{M}_s}$ is a *classical* feature of chaos. Therefore, we conclude that $\mathcal{R}_{\mathcal{M}_s} = -l$, $J_{\mathcal{M}_s} \overset{\tau\to\infty}{\sim} l\exp(\lambda\tau)$ and $\mathcal{S}_{\mathcal{M}_s} \overset{\tau\to\infty}{\sim} l\lambda\tau$. Thus, $\mathcal{R}_{\mathcal{M}_s}$, $\mathcal{S}_{\mathcal{M}_s}$ and $J_{\mathcal{M}_s}$ behave as proper indicators of chaoticity and are proportional to the number of Gaussian-distributed microstates of the system. This proportionality, even though proven in a very special case, leads to conclude there may be a substantial link among these information geometric indicators of chaoticity.

Gaussian statistical models and correlations

In [35], we apply the IGAC to study the information constrained dynamics of a system with $l = 2$ microscopic degrees of freedom. As working hypothesis, we assume that such degrees of freedom are represented by two correlated Gaussian-distributed microvariables characterized by the same variance. We show that the presence of microcorrelations lead to the emergence of an asymptotic information geometric compression of the statistical macrostates explored by the system at a faster rate than that observed in absence of microcorrelations. This result constitutes an important and explicit connection between micro-correlations and macro-complexity in statistical dynamical systems. The relevance of our finding is twofold: first, it provides a neat description of the effect of information encoded in microscopic variables on experimentally observable quantities defined in terms of dynamical macroscopic variables; second, it clearly shows the change in behavior of the macroscopic complexity of a statistical model caused by the existence of correlations at the underlying microscopic level.

Random frequency macroscopic IHOs

The problem of General Relativity is twofold: one is how geometry evolves, and the other is how particles move in a given geometry. The IGAC focuses on how particles move in a given geometry and neglects the other problem, the evolution of the geometry. The realization that there exist two separate and distinct problems was a turning point in our research and lead to an unexpected result. In [20], we explore the possibility of using well established principles of inference to derive Newtonian dynamics from relevant prior information codified into an appropriate statistical manifold. The basic assumption is that there is an irreducible uncertainty in the location of particles so that the state of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold the geometry of which is defined by the Fisher-Rao information metric. The trajectory follows from a principle of inference, the MrE method. There is no need for additional physical postulates such as an action principle or equation of motion, nor for the concept of mass, momentum and of phase space, not even the notion of time. The resulting entropic dynamics reproduces Newton's mechanics for any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained as a consequence of the underlying statistical manifold.

Following this line of reasoning, in [36], [37] we present an information geometric analogue of the Zurek-Paz quantum chaos criterion in the *classical reversible limit*. This analogy is illustrated by applying the IGAC to a set of n-uncoupled three-dimensional anisotropic inverted harmonic oscillators (IHOs) characterized by a Ohmic distributed frequency spectrum.

Regular and chaotic quantum spin chains

In [38], [39], we study the entropic dynamics on curved statistical manifolds induced by classical probability distributions of common use in the study of regular and chaotic quantum energy level statistics. Specifically, we propose an information geometric characterization of chaotic (integrable) energy level statistics of a quantum antiferromagnetic Ising spin chain in a tilted (transverse) external magnetic field. We consider the IGAC of a Poisson distribution coupled to an Exponential bath (spin chain in a *transverse* magnetic field, regular case) and that of a Wigner-Dyson distribution coupled to a Gaussian bath (spin chain in a *tilted* magnetic field, chaotic case). Remarkably, we show that in the former case the IGE exhibits asymptotic logarithmic growth while in the latter case the IGE exhibits asymptotic linear growth. In view of these findings, we conjecture our IGAC might find some potential physical applications in quantum energy level statistics as well.

Complexity reduction and statistical embedding

In [40], we characterize the complexity of geodesic paths on a curved statistical manifold $\mathcal{M}_s$ through the asymptotic computation of the IGC and the Jacobi vector field intensity $J_{\mathcal{M}_s}$. The manifold $\mathcal{M}_s$ is a $2l$-dimensional Gaussian model reproduced by an appropriate embedding in a larger $4l$-dimensional Gaussian manifold and endowed with a Fisher-Rao information metric $g_{\mu\nu}(\theta)$ with non-trivial off diagonal terms. These terms emerge due to the presence of a correlational structure (embedding constraints) among the statistical variables on the larger manifold and are characterized by macroscopic correlational coefficients r_k. First, we observe a power law decay of the information geometric complexity at a rate determined by the coefficients r_k and conclude that the non-trivial off diagonal terms lead to the emergence of an asymptotic information geometric compression of the explored macrostates on $\mathcal{M}_s$. Finally, we also observe that the presence of such embedding constraints leads to an attenuation of the asymptotic exponential divergence of the Jacobi vector field intensity. We are confident the work presented in [40] constitutes a further non-trivial step towards the characterization of the complexity of microscopically correlated multi-dimensional Gaussian statistical models, and other models of relevance in realistic physical systems.

Scattering induced quantum entanglement

In [41], [42], we present an information geometric analysis of entanglement generated by s-wave scattering between two Gaussian wave packets. We conjecture that the pre and post-collisional quantum dynamical scenarios related to an elastic head-on collision are macroscopic manifestations emerging from microscopic statistical structures. We then describe them by uncorrelated and correlated Gaussian statistical models, respectively. This allows us to express the entanglement strength in terms of scattering potential and incident particle energies. Furthermore, we show how the entanglement duration can be related to the scattering potential and incident particle energies. Finally, we discuss the connection between entanglement and complexity of motion. We are confident that the work presented in [41], [42] represents significant progress toward the goal of understanding the relationship between statistical microcorrelations and quantum entanglement on the one hand and the effect of microcorrelations on the complexity of informational geodesic flows on the other. It is also our hope to build upon the techniques employed in this work to ultimately establish a sound information geometric interpretation of quantum entanglement together with its connection to complexity of motion in more general physical scenarios.

Suppression of classical chaos and quantization

In [43], we study the information geometry and the entropic dynamics of a $3d$ Gaussian statistical model. We then compare our analysis to that of a $2d$ Gaussian statistical model obtained from the higher-dimensional model via introduction of an additional information constraint that resembles the quantum mechanical canonical minimum uncertainty relation. We show that the chaoticity (temporal complexity) of the $2d$ Gaussian statistical model, quantified by means of the IGE and the Jacobi vector field intensity, is softened with respect to the chaoticity of the $3d$ Gaussian statistical model. In view of the similarity between the information constraint on the variances and the phase-space coarse-graining imposed by the Heisenberg uncertainty relations, we suggest that our work provides a possible way of explaining the phenomenon of suppression of classical chaos operated by quantization.

In the same vein of our work in [43], a recent investigation claims that quantum mechanics can reduce the statistical complexity of classical models [44]. Specifically, it was shown that mathematical models featuring quantum effects can be as predictive as classical models although implemented by simulators that require less memory, that is, less statistical complexity. Of course, these two works use different definitions of complexity and their ultimate goal is definitively not the same. However, it is remarkable that both of them exploit some quantum feature, Heisenberg's uncertainty principle in [43] and the quantum state discrimination (information storage) method in [44], to exhibit the complexity softening effects.

Is there any link between Heisenberg's uncertainty principle and quantum state discrimination? Recently, it was shown that any violation of uncertainty relations in quantum mechanics also leads to a violation of the second law of thermodynamics [45]. In addition, it was reported in [46] that a violation of Heisenberg's uncertainty principle allows perfect state discrimination of nonorthogonal states which, in turn, violates the second law of thermodynamics [47]. The possibility of distinguishing nonorthogonal states is directly related to the question of how much information we can store in a quantum state. Information storage and memory are key quantities for the characterization of statistical complexity. In view of these considerations, it would be worthwhile exploring the possible thermodynamic link underlying these two different complexity measures.

CLOSING REMARKS

In this Contribution, we presented our information geometric measure of complexity of geodesic paths on curved statistical manifolds underlying the entropic dynamics of classical physical systems described by probability distributions within the IGAC framework. We also provided several illustrative examples of entropic dynamical models used to infer macroscopic predictions when only partial knowledge of the microscopic nature of the system is available. Finally, among other things, we also presented entropic arguments to briefly address complexity softening effects due to statistical embedding procedures.

All too often that which is correct is not new and that which is new is not correct. Being moderately conservative people, we hope that what we presented satisfies at least of one these two sub-optimal situations. We are aware that several issues remain unsolved within the IGAC framework and much more work remains to be done. However, we are immensely gratified that our scientific vision is gaining more attention and is becoming a source of inspiration for other researchers [48].

To conclude, we would like to outline the three possible lines of research for future investigations:

- Extend the IGAC to a fully quantum setting where density matrices play the analogous role of the classical probability distributions: since quantum computation can be viewed as geometry [49], [50] and computational tasks have, in general, a thermodynamic cost [51], we might envision a *thermodynamics of quantum information geometric flows on manifolds of density operators* whose ultimate internal consistency check forbids the prediction of the impossible thermodynamic machine.

- Understand the role of thermodynamics as the possible bridge among different complexity measures: softening effects in the classical-to-quantum transitions can occur provided that the various quantum effects being exploited by the different complexity measures do not violate the second law of thermodynamics;

- Describe and understand the role of thermodynamics within the IGAC: thermodynamics plays a prominent role in the entropic analysis of chaotic dynamics [52]. Chaoticity and entropic arguments are the bread and butter of the IGAC. Furthermore, inspired by [53], we could investigate the possible connection between thermodynamics inefficiency measured by dissipation and ineffectiveness of entropic dynamical models in making reliable macroscopic predictions.

ACKNOWLEDGMENT

I acknowledge that this work reflects my academic interaction with the following scientists: Erik Bollt (Potsdam-NY, USA), Carlo Bradaschia (Pisa, Italy), Ariel Caticha (Albany-NY, USA), Giancarlo Cella (Pisa, Italy), Stefano Mancini (Camerino, Italy), Jie Sun (Potsdam-NY, USA), Peter van Loock (Erlangen and Mainz, Germany). I also thank Sean Alan Ali and Adom Giffin whose careful reviews strengthened this Contribution and Denis Gonta for technical assistance. Finally, special thanks go to Gianpaolo Beretta for inviting me as an Invited Speaker at *The 12th Biannual Joint European Thermodynamics Conference* in Brescia, Italy.

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